

# SWEEPING PROCESSES PERTURBED BY ROUGH SIGNALS

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**ABSTRACT.** This paper deals with the existence of solutions to sweeping processes perturbed by a continuous signal of finite  $p$ -variation with  $p \in [1, 3]$ . It covers pathwise stochastic multiplicative noises directed by a fractional Brownian motion of Hurst parameter greater than  $1/3$ . The rough integral and a continuity result established by C. Castaing et al. in [6] are the cornerstone of this work.

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## 1. INTRODUCTION

Consider a multifunction  $C : [0, T] \rightrightarrows \mathbb{R}^e$  with  $e \in \mathbb{N}^*$ . Roughly speaking, the Moreau sweeping process (see Moreau [19]) associated to  $C$  is the path  $X$ , living in  $C$ , such that when it hits the frontier of  $C$ , a minimal force is applied to  $X$  in order to keep it inside of  $C$ . Precisely,  $X$  is the solution to the following differential inclusion:

$$(1) \quad \begin{cases} -\frac{dDY}{d|DY|}(t) \in N_{C(t)}(Y(t)) \text{ } |DY|\text{-a.e.} \\ Y(0) = a \in C(0), \end{cases}$$

where  $DY$  is the differential measure associated with the continuous function of bounded variation  $Y$ ,  $|DY|$  is its variation measure, and  $N_{C(t)}(Y(t))$  is the normal cone of  $C(t)$  at  $Y(t)$ . This problem has been deeply studied by many authors. For instance, the reader can refer to Moreau [19], Valadier [21] or Monteiro Marques [18].

Several authors studied some perturbed versions of Problem (1), in particular by a stochastic multiplicative noise in Itô's calculus framework (see Revuz and Yor [20]). For instance, the reader can refer to Bernicot and Venel [3] or Castaing et al. [6]. On reflected diffusion processes, which are related to perturbed sweeping processes, the reader can refer to Kang and Ramanan [13].

Consider the perturbed Skorokhod problem

$$(2) \quad \begin{cases} X(t) = H(t) + Y(t) \\ H(t) = \int_0^t f(X(s)) dZ(s) \\ -\frac{dDY}{d|DY|}(t) \in N_{C_H(t)}(Y(t)) \text{ } |DY|\text{-a.e. with } Y(0) = a \end{cases}$$

where,  $C_H(t) = C(t) - H(t)$ ,  $t \in [0, T]$  (thus  $N_{C_H(t)}(Y(t)) = N_{C(t)}(X(t))$ ),  $Z : [0, T] \rightarrow \mathbb{R}^d$  is a continuous signal of finite  $p$ -variation with  $d \in \mathbb{N}^*$  and  $p \in [1, \infty[$ ,  $f \in \text{Lip}^\gamma(\mathbb{R}^e, \mathcal{M}_{e,d}(\mathbb{R}))$  with  $\gamma > p$ , and the integral against  $Z$  is taken in the sense of rough paths. On the rough integral, the reader can refer to Lyons [15], Friz and Victoir [11] or Friz and Hairer [9]. Throughout the paper, the multifunction  $C$  satisfies the following assumption.

**Assumption 1.1.**  *$C$  is a convex compact valued multifunction, continuous for the Hausdorff distance, and there exists a continuous selection  $\gamma : [0, T] \rightarrow \mathbb{R}^e$  satisfying*

$$\overline{B}_e(\gamma(t), r) \subset C(t) ; \forall t \in [0, T],$$

where  $\overline{B}_e(\gamma(t), r)$  denotes the closed ball of radius  $r$  centered at  $\gamma(t)$ .

This assumption is equivalent to saying that  $C(t)$  has nonempty interior for every  $t \in [0, T]$ , see [6, Lemma 2.2].

In Falkowski and Slomiński [8], when  $p \in [1, 2[$  and  $C(t)$  is a cuboid of  $\mathbb{R}^e$  for every  $t \in [0, T]$ , the authors proved the existence and uniqueness of the solution of Problem (1). Furthermore, several authors studied the existence and uniqueness of the solution for reflected rough differential equations. In [2], M. Besalú et al. proved the existence and uniqueness of the solution for delayed rough differential equations with non-negativity constraints. Recently, S. Aida gets the existence of solutions for a large class of reflected rough differential equations in [1] and [2]. Finally, in [7], A. Deya et al. proved the existence and uniqueness of the solution for 1-dimensional reflected rough differential equations. An interesting remark related to these references is that when  $C$  is not a cuboid, moving or not, it is a challenge to get the uniqueness of the solution for reflected rough differential equations and sweeping processes.

The purpose of this paper is to prove the existence of solutions for Problem (1) when  $p \in [1, 3[$  and  $C$  satisfies Assumption 1.1.

Section 2 deals with some preliminaries on sweeping processes and the rough integral. Section 3 is devoted to the existence of solutions to Problem (1) when  $Z$  is a moderately irregular signal (i.e.  $p \in [1, 2[$ ). For instance,  $Z$  could be a path of a fractional Brownian motion of Hurst parameter belonging to  $]1/2, 1[$ . Section 4 deals with the existence of solutions to Problem (1) when  $Z$  is a rough signal (i.e.  $p \in [2, 3[$ ). For instance,  $Z$  could be a path of a fractional Brownian motion of Hurst parameter belonging to  $]1/3, 1/2[$ .

The following notations, definitions and properties are used throughout the paper.

#### Notations and elementary properties:

1.  $C_h(t) := C(t) - h(t)$  for every function  $h : [0, T] \rightarrow \mathbb{R}^e$ .
2.  $N_C(x)$  is the normal cone of  $C$  at  $x$ , for any closed convex subset  $C$  of  $\mathbb{R}^e$  and any  $x \in \mathbb{R}^e$  (recall that  $N_C(x) = \emptyset$  if  $x \notin C$ ).

3.  $\Delta_T := \{(s, t) \in [0, T]^2 : s < t\}$  and  $\Delta_{s,t} := \{(u, v) \in [s, t]^2 : u < v\}$  for every  $(s, t) \in \Delta_T$ .
4. For every function  $x$  from  $[0, T]$  into  $\mathbb{R}^d$  and  $(s, t) \in \Delta_T$ ,  $x(s, t) := x(t) - x(s)$ .
5. Consider  $(s, t) \in \Delta_T$ . The vector space of continuous functions from  $[s, t]$  into  $\mathbb{R}^d$  is denoted by  $C^0([s, t], \mathbb{R}^d)$  and equipped with the uniform norm  $\|\cdot\|_{\infty, s, t}$  defined by

$$\|x\|_{\infty, s, t} := \sup_{u \in [s, t]} \|x(u)\|$$

for every  $x \in C^0([s, t], \mathbb{R}^d)$ . Moreover,  $\|\cdot\|_{\infty, T} := \|\cdot\|_{\infty, 0, T}$  and

$$C_0^0([s, t], \mathbb{R}^d) := \{x \in C^0([s, t], \mathbb{R}^d) : x(0) = 0\}.$$

6. Consider  $(s, t) \in \Delta_T$ . The set of all dissections of  $[s, t]$  is denoted by  $\mathfrak{D}_{[s, t]}$ .
7. Consider  $(s, t) \in \Delta_T$ . A continuous function  $x : [s, t] \rightarrow \mathbb{R}^d$  is of finite  $p$ -variation if and only if,

$$\|x\|_{p\text{-var}, s, t} := \sup \left\{ \left| \sum_{k=1}^{n-1} \|x(t_k, t_{k+1})\|^p \right|^{1/p} ; n \in \mathbb{N}^* \text{ and } (t_k)_{k \in \llbracket 1, n \rrbracket} \in \mathfrak{D}_{[s, t]} \right\} < \infty.$$

Consider the vector space

$$C^{p\text{-var}}([s, t], \mathbb{R}^d) := \{x \in C^0([s, t], \mathbb{R}^d) : \|x\|_{p\text{-var}, s, t} < \infty\}.$$

The map  $\|\cdot\|_{p\text{-var}, s, t}$  is a semi-norm on  $C^{p\text{-var}}([s, t], \mathbb{R}^d)$ .

Moreover,  $\|\cdot\|_{p\text{-var}, T} := \|\cdot\|_{p\text{-var}, 0, T}$ .

*Remarks :*

- a. Note that for every  $q, r \in [1, \infty[$  such that  $q \geq r$ ,

$$\forall x \in C^{r\text{-var}}([s, t], \mathbb{R}^d), \|x\|_{q\text{-var}, s, t} \leq \|x\|_{r\text{-var}, s, t}.$$

In particular, any continuous function of bounded variation on  $[s, t]$  belongs to  $C^{q\text{-var}}([s, t], \mathbb{R}^d)$  for every  $q \in [1, \infty[$ .

- b. Note that for every  $(s, t) \in \Delta_T$  and  $x \in C^{1\text{-var}}([s, t], \mathbb{R})$ ,

$$\|x\|_{1\text{-var}, s, t} = \int_s^t |Dx|$$

where,  $|Dy|$  is the variation measure of the differential measure  $Dx$  associated with  $x$ .

8. The vector space of Lipschitz continuous maps from  $\mathbb{R}^e$  into  $\mathcal{M}_{e,d}(\mathbb{R})$  is denoted by  $\text{Lip}(\mathbb{R}^e, \mathcal{M}_{e,d}(\mathbb{R}))$  and equipped with the Lipschitz semi-norm  $\|\cdot\|_{\text{Lip}}$  defined by

$$\|\varphi\|_{\text{Lip}} := \sup \left\{ \frac{\|\varphi(y) - \varphi(x)\|}{\|y - x\|} ; x, y \in \mathbb{R}^e \text{ and } x \neq y \right\}$$

for every  $\varphi \in \text{Lip}(\mathbb{R}^e, \mathcal{M}_{e,d}(\mathbb{R}))$ .

9. For every  $\lambda \in \mathbb{R}$ ,

$$\lfloor \lambda \rfloor := \max\{n \in \mathbb{Z} : n < \lambda\}$$

and  $\{\lambda\} := \lambda - \lfloor \lambda \rfloor$ .

10. Consider  $\gamma \in [1, \infty[$ . A continuous map  $\varphi : \mathbb{R}^e \rightarrow \mathcal{M}_{d,e}(\mathbb{R})$  is  $\gamma$ -Lipschitz in the sense of Stein if and only if,

$$\|\varphi\|_{\text{Lip}^\gamma} := \|D^{\lfloor \gamma \rfloor} \varphi\|_{\{\gamma\}\text{-Höl}} \vee \max\{\|D^k \varphi\|_\infty ; k \in \llbracket 0, \lfloor \gamma \rfloor \rrbracket\} < \infty.$$

Consider the vector space

$$\text{Lip}^\gamma(\mathbb{R}^e, \mathcal{M}_{e,d}(\mathbb{R})) := \{\varphi \in C^0(\mathbb{R}^e, \mathcal{M}_{e,d}(\mathbb{R})) : \|\varphi\|_{\text{Lip}^\gamma} < \infty\}.$$

The map  $\|\cdot\|_{\text{Lip}^\gamma}$  is a norm on  $\text{Lip}^\gamma(\mathbb{R}^e, \mathcal{M}_{e,d}(\mathbb{R}))$ .

*Remarks:*

- a. If  $\varphi \in \text{Lip}^\gamma(\mathbb{R}^e, \mathcal{M}_{e,d}(\mathbb{R}))$ , then  $\varphi \in \text{Lip}(\mathbb{R}^e, \mathcal{M}_{e,d}(\mathbb{R}))$ .
- b. If  $\varphi \in C^{\lfloor \gamma \rfloor + 1}(\mathbb{R}^e, \mathcal{M}_{e,d}(\mathbb{R}))$  is bounded with bounded derivatives, then  $\varphi \in \text{Lip}^\gamma(\mathbb{R}^e, \mathcal{M}_{e,d}(\mathbb{R}))$ .

## 2. PRELIMINARIES

This section deals with some preliminaries on sweeping processes and the rough integral. The first subsection states some fundamental results on unperturbed sweeping processes coming from Moreau [19], Valadier [21] and Monteiro Marques [18]. A continuity result of Castaing et al. [6], which is the cornerstone of the proofs of Theorem 3.1 and Theorem 4.1, is also stated. The second subsection deals with the integration along rough paths. In this paper, definitions and propositions are stated as in Friz and Hairer [9], in accordance with M. Gubinelli's approach (see Gubinelli [12]).

**2.1. Sweeping processes.** The following theorem, due to Monteiro Marques [16, 17, 18] using an estimation due to Valadier (see [5, 21]), states a sufficient condition of existence and uniqueness of the solution of the unperturbed sweeping process defined by Problem (1).

**Proposition 2.1.** *Assume that  $C$  is a convex compact valued multifunction, continuous for the Hausdorff distance, and such that there exists  $(x, r) \in \mathbb{R}^e \times ]0, \infty[$  satisfying*

$$\overline{B}_e(x, r) \subset C(t) ; \forall t \in [0, T].$$

*Then, Problem (1) has a unique continuous solution of finite 1-variation  $y : [0, T] \rightarrow \mathbb{R}^e$  such that*

$$\|y\|_{1\text{-var}, T} \leq \mathfrak{l}(r, \|a - x\|)$$

where,  $\mathfrak{l} : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+$  is the map defined by

$$\mathfrak{l}(s, S) := \begin{cases} \max \left\{ 0, \frac{S^2 - s^2}{2s} \right\} & \text{if } e > 1 \\ \max\{0, S - s\} & \text{if } e = 1 \end{cases} ; \forall s, S \in \mathbb{R}_+.$$

See Monteiro Marques [18] for a proof.

Let  $h$  be a continuous function from  $[0, T]$  into  $\mathbb{R}^e$  such that  $h(0) = 0$ . If it exists, a Skorokhod decomposition of  $(C, a, h)$  is a couple  $(v_h, w_h)$  such that:

$$(3) \quad \begin{cases} v_h(t) = h(t) + w_h(t) \\ -\frac{dDw_h}{d|Dw_h|}(t) \in N_{C_h(t)}(w_h(t)) \text{ with } w_h(0) = a. \end{cases}$$

Under Assumption 1.1, by Proposition 2.1 together with Castaing et al. [6, Lemma 2.2],  $(C, a, h)$  has a unique Skorokhod decomposition  $(v_h, w_h)$ .

**Theorem 2.2.** *Under Assumption 1.1, if  $(h_n)_{n \in \mathbb{N}}$  is a sequence of continuous functions from  $[0, T]$  into  $\mathbb{R}^e$  which converges uniformly to  $h \in C^0([0, T], \mathbb{R}^e)$ , then*

$$\sup_{n \in \mathbb{N}} \|w_{h_n}\|_{1\text{-var}, T} < \infty$$

and

$$(v_{h_n}, w_{h_n}) \xrightarrow[n \rightarrow \infty]{\|\cdot\|_{\infty, T}} (v_h, w_h).$$

See Castaing et al. [6, Theorem 2.3].

**Proposition 2.3.** *Under Assumption 1.1:*

(1) *The map  $(v, w)$  is continuous from*

$$C_0^0([0, T], \mathbb{R}^e) \text{ to } C^0([0, T], \mathbb{R}^e) \times C^{1\text{-var}}([0, T], \mathbb{R}^e).$$

(2) *For every  $(s, t) \in \Delta_T$ ,*

$$\|w_h\|_{1\text{-var}, s, t} \leq \sup_{u \in [s, t]} \mathbf{I}(r, \|\gamma(u) - h(u)\|).$$

See Castaing et al. [6, Lemma 5.3] for a proof.

**2.2. Young's integral, rough integral.** The first part of the subsection deals with the definition and some basic properties of Young's integral which allow to integrate a map  $y \in C^{r\text{-var}}([0, T], \mathcal{M}_{e,d}(\mathbb{R}))$  with respect to  $z \in C^{q\text{-var}}([0, T], \mathbb{R}^d)$  when  $q, r \in [1, \infty[$  and  $1/q + 1/r > 1$ . The second part of the subsection deals with the rough integral which extends Young's integral when the condition  $1/q + 1/r > 1$  is not satisfied anymore. The signal  $z$  has to be enhanced as a rough path.

**Definition 2.4.** *A map  $\omega : \Delta_T \rightarrow \mathbb{R}_+$  is a control function if and only if,*

- (1)  *$\omega$  is continuous.*
- (2)  *$\omega(s, s) = 0$  for every  $s \in [0, T]$ .*
- (3)  *$\omega$  is super-additive:*

$$\omega(s, u) + \omega(u, t) \leq \omega(s, t)$$

*for every  $s, t, u \in [0, T]$  such that  $s \leq u \leq t$ .*

**Example.** For every  $z \in C^{p\text{-var}}([0, T], \mathbb{R}^d)$ , the map

$$\omega_z : (s, t) \in \Delta_T \mapsto \omega_z(s, t) := \|z\|_{p\text{-var}, s, t}^p$$

is a control function.

**Proposition 2.5.** *Consider  $x \in C^0([0, T], \mathbb{R}^d)$  and a sequence  $(x_n)_{n \in \mathbb{N}}$  of elements of  $C^{p\text{-var}}([0, T], \mathbb{R}^d)$  such that:*

$$\lim_{n \rightarrow \infty} \|x_n - x\|_{\infty, T} = 0 \text{ and } \sup_{n \in \mathbb{N}} \|x_n\|_{p\text{-var}, T} < \infty.$$

*Then,  $x \in C^{p\text{-var}}([0, T], \mathbb{R}^d)$  and*

$$\lim_{n \rightarrow \infty} \|x_n - x\|_{(p+\varepsilon)\text{-var}, T} = 0 ; \forall \varepsilon > 0.$$

See Friz and Victoir [11, Lemma 5.12 and Lemma 5.27] for a proof.

**Proposition 2.6.** *(Young's integral) Consider  $q, r \in [1, \infty[$  such that  $1/q + 1/r > 1$ , and two maps  $y \in C^{r\text{-var}}([0, T], \mathcal{M}_{e,d}(\mathbb{R}))$  and  $z \in C^{q\text{-var}}([0, T], \mathbb{R}^d)$ . For every  $n \in \mathbb{N}^*$  and  $(t_k^n)_{k \in \llbracket 1, n \rrbracket} \in \mathfrak{D}_{[0, T]}$ , the limit*

$$\lim_{n \rightarrow \infty} \sum_{k=1}^{n-1} y(t_k^n) z(t_k^n, t_{k+1}^n)$$

*exists and does not depend on the dissection  $(t_k^n)_{k \in \llbracket 1, n \rrbracket}$ . That limit is denoted by*

$$\int_0^T y(s) dz(s)$$

*and called Young's integral of  $y$  with respect to  $z$  on  $[0, T]$ . Moreover, there exists a constant  $c(q, r) > 0$ , depending only on  $q$  and  $r$ , such that for every  $(s, t) \in \Delta_T$ ,*

$$\left\| \int_0^\cdot y(s) dz(s) \right\|_{r\text{-var}, s, t} \leq c(q, r) \|z\|_{q\text{-var}, s, t} (\|y\|_{r\text{-var}, s, t} + \|y\|_{\infty, s, t}).$$

See Lyons [15, Theorem 1.16], Lejay [14, Theorem 1] or Friz and Victoir [11, Theorem 6.8] for a proof.

**Proposition 2.7.** *Consider  $q, r \in [1, \infty[$  such that  $1/q + 1/r > 1$ , two maps  $y \in C^{r\text{-var}}([0, T], \mathcal{M}_{e,d}(\mathbb{R}))$  and  $z \in C^{q\text{-var}}([0, T], \mathbb{R}^d)$ , and a sequence  $(y_n)_{n \in \mathbb{N}}$  of elements of  $C^{r\text{-var}}([0, T], \mathcal{M}_{e,d}(\mathbb{R}))$  such that:*

$$\lim_{n \rightarrow \infty} \|y_n - y\|_{\infty, T} = 0 \text{ and } \sup_{n \in \mathbb{N}} \|y_n\|_{r\text{-var}, T} < \infty.$$

Then,

$$\lim_{n \rightarrow \infty} \left\| \int_0^\cdot y_n(s) dz(s) - \int_0^\cdot y(s) dz(s) \right\|_{\infty, T} = 0.$$

See Friz and Victoir [11, Proposition 6.12] for a proof.

Consider  $p \in [2, 3[$  and let us define the rough integral for continuous functions of finite  $p$ -variation.

**Remark.** In the sequel, the reader has to keep in mind that:

- (1)  $\mathcal{M}_{e,d}(\mathbb{R}) \cong \mathbb{R}^e \otimes \mathbb{R}^d$ .
- (2)  $\mathcal{M}_{d,1}(\mathbb{R}) \cong \mathcal{M}_{1,d}(\mathbb{R}) \cong \mathbb{R}^d$ .
- (3)  $\mathcal{L}(\mathbb{R}^d, \mathcal{M}_{e,d}(\mathbb{R})) \cong \mathcal{L}(\mathbb{R}^d, \mathcal{L}(\mathbb{R}^d, \mathbb{R}^e)) \cong \mathcal{L}(\mathbb{R}^d \otimes \mathbb{R}^d, \mathbb{R}^e)$ .

**Definition 2.8.** *Consider  $z \in C^{1\text{-var}}([0, T], \mathbb{R}^d)$ . The step-2 signature of  $z$  is the map  $S_2(z) : \Delta_T \rightarrow \mathbb{R}^d \times \mathcal{M}_d(\mathbb{R})$  defined by*

$$S_2(z)(s, t) := \left( z(s, t), \int_{s < u < v < t} dz(v) \otimes dz(u) \right)$$

for every  $(s, t) \in \Delta_T$ .

**Notation.**  $\mathfrak{S}_T(\mathbb{R}^d) := \{S_2(z)(0, \cdot) ; z \in C^{1\text{-var}}([0, T], \mathbb{R}^d)\}$ .

**Definition 2.9.** *The geometric  $p$ -rough paths metric space  $G\Omega_{p,T}(\mathbb{R}^d)$  is the closure of  $\mathfrak{S}_T(\mathbb{R}^d)$  in  $C^{p\text{-var}}([0, T], \mathbb{R}^d) \times C^{p/2\text{-var}}([0, T], \mathcal{M}_d(\mathbb{R}))$ .*

**Definition 2.10.** *For  $z \in C^{p\text{-var}}([0, T], \mathbb{R}^d)$ , a map  $y \in C^{p\text{-var}}([0, T], \mathcal{M}_{e,d}(\mathbb{R}))$  is controlled by  $z$  if and only if there exists  $y' \in C^{p\text{-var}}([0, T], \mathcal{L}(\mathbb{R}^d, \mathcal{M}_{e,d}(\mathbb{R})))$  such that*

$$y(s, t) = y'(s)z(s, t) + R_y(s, t) ; \forall (s, t) \in \Delta_T$$

with  $\|R_y\|_{p/2\text{-var}, T} < \infty$ . For fixed  $z$ , the pairs  $(y, y')$  as above define a vector space denoted by  $\mathfrak{D}_z^{p/2}([0, T], \mathcal{M}_{e,d}(\mathbb{R}))$  and equipped with the semi-norm  $\|\cdot\|_{z, p/2, T}$  such that

$$\|(y, y')\|_{z, p/2, T} := \|y'\|_{p\text{-var}, T} + \|R_y\|_{p/2\text{-var}, T}$$

for every  $(y, y') \in \mathfrak{D}_z^{p/2}([0, T], \mathcal{M}_{e,d}(\mathbb{R}))$ .

**Theorem 2.11.** *(Rough integral) Consider  $\mathbf{z} := (z, \mathbb{Z}) \in G\Omega_{p,T}(\mathbb{R}^d)$  and  $(y, y') \in \mathfrak{D}_z^{p/2}([0, T], \mathcal{M}_{e,d}(\mathbb{R}))$ . For every  $n \in \mathbb{N}^*$  and  $(t_k^n)_{k \in [1, n]} \in \mathfrak{D}_{[0, T]}$ , the limit*

$$\lim_{n \rightarrow \infty} \sum_{k=1}^{n-1} (y(t_k^n)z(t_k^n, t_{k+1}^n) + y'(t_k^n)\mathbb{Z}(t_k^n, t_{k+1}^n))$$

exists and does not depend on the dissection  $(t_k^n)_{k \in [1, n]}$ . That limit is denoted by

$$\int_0^T y(s) d\mathbf{z}(s)$$

and called rough integral of  $y$  with respect to  $\mathbf{z}$  on  $[0, T]$ . Moreover,

(1) There exists a constant  $c(p) > 0$ , depending only on  $p$ , such that for every  $(s, t) \in \Delta_T$ ,

$$\left\| \int_s^t y(u) d\mathbf{z}(u) - y(s)z(s, t) - y'(s)\mathbb{Z}(s, t) \right\| \leq c(p)(\|z\|_{p\text{-var}, s, t} \|R_y\|_{p/2\text{-var}, s, t} + \|\mathbb{Z}\|_{p/2\text{-var}, s, t} \|y'\|_{p\text{-var}, s, t}).$$

(2) The map

$$(y, y') \mapsto \left( \int_0^\cdot y(s) d\mathbf{z}(s), y \right)$$

is continuous from  $\mathfrak{D}_z^{p/2}([0, T], \mathcal{M}_{e,d}(\mathbb{R}))$  into  $\mathfrak{D}_z^{p/2}([0, T], \mathbb{R}^e)$ .

See Friz and Shekhar [10, Theorem 34] for a proof with the  $p$ -variation topology, and see Gubinelli [12, Theorem 1] or Friz and Hairer [9, Theorem 4.10] for a proof with the  $1/p$ -Hölder topology.

**Proposition 2.12.** Consider  $\mathbf{z} := (z, \mathbb{Z}) \in G\Omega_{p,T}(\mathbb{R}^d)$ , a continuous map

$$(y, y') : [0, T] \longrightarrow \mathcal{M}_{e,d}(\mathbb{R}) \times \mathcal{L}(\mathbb{R}^d, \mathcal{M}_{e,d}(\mathbb{R})),$$

and a sequence  $(y_n, y'_n)_{n \in \mathbb{N}}$  of elements of  $\mathfrak{D}_z^{p/2}([0, T], \mathcal{M}_{e,d}(\mathbb{R}))$  such that

$$(y'_n, R_{y_n}) \xrightarrow[n \rightarrow \infty]{d_{\infty, T}} (y', R_y) \text{ and } \sup_{n \in \mathbb{N}} \|(y_n, y'_n)\|_{z, p/2, T} < \infty.$$

Then,  $(y, y') \in \mathfrak{D}_z^{p/2}([0, T], \mathcal{M}_{e,d}(\mathbb{R}))$  and

$$\lim_{n \rightarrow \infty} \left\| \int_0^\cdot y_n(s) d\mathbf{z}(s) - \int_0^\cdot y(s) d\mathbf{z}(s) \right\|_{\infty, T} = 0.$$

*Proof.* On the one hand, since

$$(y'_n, R_{y_n}) \xrightarrow[n \rightarrow \infty]{d_{\infty, T}} (y', R_y),$$

the function  $y$  is the uniform limit of the sequence  $(y_n)_{n \in \mathbb{N}}$ . Moreover, since

$$\sup_{n \in \mathbb{N}} \|(y_n, y'_n)\|_{z, p/2, T} < \infty,$$

by Proposition 2.5,

$$y' \in C^{p\text{-var}}([0, T], \mathcal{L}(\mathbb{R}^d, \mathcal{M}_{e,d}(\mathbb{R}))) \text{ and } R_y \in C^{p/2\text{-var}}([0, T], \mathcal{M}_{e,d}(\mathbb{R})).$$

So,  $(y, y') \in \mathfrak{D}_z^{p/2}([0, T], \mathcal{M}_{e,d}(\mathbb{R}))$ .

On the other hand, also by Proposition 2.5, for any  $\varepsilon > 0$  such that  $p + \varepsilon \in [2, 3[$ ,

$$\begin{aligned} \lim_{n \rightarrow \infty} \|(y_n, y'_n) - (y, y')\|_{z, (p+\varepsilon)/2, T} &= \lim_{n \rightarrow \infty} \|y'_n - y'\|_{(p+\varepsilon)\text{-var}, T} + \\ &\quad \lim_{n \rightarrow \infty} \|R_{y_n} - R_y\|_{(p+\varepsilon)/2\text{-var}, T} \\ &= 0. \end{aligned}$$

So, by continuity of the rough integral (see Theorem 2.11),

$$\lim_{n \rightarrow \infty} \left\| \left( \int_0^\cdot y_n(s) d\mathbf{z}(s), y_n \right) - \left( \int_0^\cdot y(s) d\mathbf{z}(s), y \right) \right\|_{z, (p+\varepsilon)/2, T} = 0.$$

Therefore, in particular:

$$\lim_{n \rightarrow \infty} \left\| \int_0^\cdot y_n(s) d\mathbf{z}(s) - \int_0^\cdot y(s) d\mathbf{z}(s) \right\|_{\infty, T} = 0.$$

□

**Proposition 2.13.** Consider  $\mathbf{z} := (z, \mathbb{Z}) \in G\Omega_{p,T}(\mathbb{R}^d)$ ,  $(x, x') \in \mathfrak{D}_z^{p/2}([0, T], \mathbb{R}^e)$  and  $\varphi \in \text{Lip}^{\gamma-1}(\mathbb{R}^e, \mathbb{R}^d)$ . The couple of maps  $(\varphi(x), \varphi(x'))$ , defined by

$$\varphi(x)(t) := \varphi(x(t)) \text{ and } \varphi(x)'(t) := D\varphi(x(t))x'(t)$$

for every  $t \in [0, T]$ , belongs to  $\mathfrak{D}_z^{p/2}([0, T], \mathcal{M}_{e,d}(\mathbb{R}))$ .

**Remark.** By Theorem 2.11 and Proposition 2.13 together,

$$\int_0^\cdot \varphi(x(u))d\mathbf{z}(u)$$

is defined. For every  $(s, t) \in \Delta_T$ , consider

$$\mathfrak{I}_{\varphi, \mathbf{z}, x}(s, t) := \left\| \int_s^t \varphi(x(u))d\mathbf{z}(u) - \varphi(x(s))z(s, t) - D\varphi(x(s))x'(s)\mathbb{Z}(s, t) \right\|.$$

For every  $(s, t) \in \Delta_T$ , since

$$\begin{aligned} \|\varphi(x)\|_{p\text{-var}, s, t} &\leq \|\varphi\|_{\text{Lip}^{\gamma-1}} \|x\|_{p\text{-var}, s, t}, \\ \|\varphi(x)'\|_{p\text{-var}, s, t} &\leq \|\varphi\|_{\text{Lip}^{\gamma-1}} (\|x'\|_{p\text{-var}, s, t} + \|x'\|_{\infty, s, t} \|x\|_{p\text{-var}, s, t}) \text{ and} \\ \|R_{\varphi(x)}\|_{p/2\text{-var}, s, t} &\leq \|\varphi\|_{\text{Lip}^{\gamma-1}} (\|x\|_{p\text{-var}, s, t}^2 + \|R_x\|_{p/2\text{-var}, s, t}), \end{aligned}$$

by Theorem 2.11,

$$\begin{aligned} \mathfrak{I}_{\varphi, \mathbf{z}, x}(s, t) &\leq c(p) \|\varphi\|_{\text{Lip}^{\gamma-1}} (\|x'\|_{p\text{-var}, s, t} + \|x'\|_{\infty, s, t} \|x\|_{p\text{-var}, s, t} + \\ &\quad \|x\|_{p\text{-var}, s, t}^2 + \|R_x\|_{p/2\text{-var}, s, t}) \omega_{\mathbf{z}}(s, t)^{1/p} \end{aligned}$$

where,  $\omega_{\mathbf{z}} : \Delta_T \rightarrow \mathbb{R}_+$  is the control function defined by

$$\omega_{\mathbf{z}}(u, v) := 2^{p-1} (\|z\|_{p\text{-var}, u, v}^p + \|\mathbb{Z}\|_{p/2\text{-var}, u, v}^p) ; \forall (u, v) \in \Delta_T.$$

**Proposition 2.14.** Consider  $\mathbf{z} := (z, \mathbb{Z}) \in G\Omega_{p,T}(\mathbb{R}^d)$ ,  $\varphi \in \text{Lip}^{\gamma-1}(\mathbb{R}^e, \mathbb{R}^d)$ , a continuous map

$$(x, x') : [0, T] \longrightarrow \mathbb{R}^e \times \mathcal{L}(\mathbb{R}^d, \mathbb{R}^e),$$

and a sequence  $(x_n, x'_n)_{n \in \mathbb{N}}$  of elements of  $\mathfrak{D}_z^{p/2}([0, T], \mathbb{R}^e)$  such that

$$(x'_n, R_{x_n}) \xrightarrow[n \rightarrow \infty]{d_{\infty, T}} (x', R_x) \text{ and } \sup_{n \in \mathbb{N}} \|(x_n, x'_n)\|_{z, p/2, T} < \infty.$$

Then,  $(\varphi(x), \varphi(x')) \in \mathfrak{D}_z^{p/2}([0, T], \mathcal{M}_{e,d}(\mathbb{R}))$  and

$$\lim_{n \rightarrow \infty} \left\| \int_0^\cdot \varphi(x_n(s))d\mathbf{z}(s) - \int_0^\cdot \varphi(x(s))d\mathbf{z}(s) \right\|_{\infty, T} = 0.$$

*Proof.* Since

$$(x'_n, R_{x_n}) \xrightarrow[n \rightarrow \infty]{d_{\infty, T}} (x', R_x) \text{ and } \sup_{n \in \mathbb{N}} \|(x_n, x'_n)\|_{z, p/2, T} < \infty,$$

by Friz and Hairer [9, Theorem 7.5] together with Proposition 2.5,

$$(\varphi(x_n)', R_{\varphi(x_n)}) \xrightarrow[n \rightarrow \infty]{d_{\infty, T}} (\varphi(x)', R_{\varphi(x)}) \text{ and } \sup_{n \in \mathbb{N}} \|(\varphi(x_n), \varphi(x_n)')\|_{z, p/2, T} < \infty.$$

So, by Proposition 2.12,

$$\lim_{n \rightarrow \infty} \left\| \int_0^\cdot \varphi(x_n(s))d\mathbf{z}(s) - \int_0^\cdot \varphi(x(s))d\mathbf{z}(s) \right\|_{\infty, T} = 0.$$

□



## 3. A SWEEPING PROCESS PERTURBED BY A MODERATELY IRREGULAR SIGNAL

Throughout this section,  $p \in [1, 2[$ . In other words, the signal  $Z$  is moderately irregular.

The existence of a solution to Problem (2) is established in Theorem 3.1. When  $p = 1$ , the uniqueness of that solution is established in Proposition 3.2.

**Theorem 3.1.** *Under Assumption 1.1, Problem (2) has at least one solution which belongs to  $C^{p\text{-var}}([0, T], \mathbb{R}^e)$ .*

*Proof.* Consider the discrete scheme

$$(4) \quad \begin{cases} X_n(t) = H_n(t) + Y_n(t) \\ H_n(t) = \int_0^t f(X_{n-1}(s)) dZ(s) \\ -\frac{dDY_n}{d|DY_n|}(t) \in N_{C_{H_n}(t)}(Y_n(t)) \text{ } |DY_n| \text{-a.e. with } Y_n(0) = a \end{cases}$$

for Problem (2), initialized by

$$(5) \quad \begin{cases} -\frac{dDX_0}{d|DX_0|}(t) \in N_{C(t)}(X_0(t)) \text{ } |DX_0| \text{-a.e.} \\ X_0(0) = a. \end{cases}$$

Since the map  $\|Z\|_{p\text{-var}, 0, \cdot}$  is continuous from  $[0, T]$  into  $\mathbb{R}_+$ , and since  $\|Z\|_{p\text{-var}, 0, 0} = 0$ , there exists  $\tau_0 \in [0, T]$  such that

$$\|Z\|_{p\text{-var}, \tau_0} \leq m := \frac{1}{c(p, p)\|f\|_{\text{Lip}^\gamma}(M+1)}$$

with  $M := 1 + c_2 + c_1$ ,  $c_1 := 1/r$  and

$$c_2 := c_1 \sup_{s \in [0, T]} \|\gamma(s)\|^2.$$

Let us show that for every  $n \in \mathbb{N}$ ,

$$(6) \quad \|X_n\|_{p\text{-var}, \tau_0} \leq M.$$

By (5) together with Proposition 2.3,

$$\begin{aligned} \|X_0\|_{p\text{-var}, \tau_0} &\leq \sup_{s \in [0, \tau_0]} \mathfrak{l}(r, \|\gamma(s)\|) \\ &\leq M. \end{aligned}$$

Assume that Inequality (6) is satisfied for  $n \in \mathbb{N}$  arbitrarily chosen. By Proposition 2.6, and since  $\|Z\|_{p\text{-var}, 0, \cdot}$  is an increasing map,

$$\begin{aligned} \|H_{n+1}\|_{p\text{-var}, \tau_0} &\leq c(p, p)\|Z\|_{p\text{-var}, \tau_0}(\|Df\|_\infty\|X_n\|_{p\text{-var}, \tau_0} + \|f \circ X_n\|_{\infty, \tau_0}) \\ &\leq mc(p, p)\|f\|_{\text{Lip}^\gamma}(M+1) \\ &\leq 1. \end{aligned}$$

Since  $Y_{n+1} = w_{H_{n+1}}$ , by Proposition 2.3,

$$\begin{aligned} \|Y_{n+1}\|_{1\text{-var}, \tau_0} &\leq \sup_{s \in [0, \tau_0]} \mathfrak{l}(r, \|\gamma(s) - H_{n+1}(s)\|) \\ &\leq c_2 + c_1 \sup_{s \in [0, \tau_0]} \|H_{n+1}(s)\|^2 \\ &\leq c_2 + c_1 \|H_{n+1}\|_{p\text{-var}, \tau_0}^2. \end{aligned}$$

Therefore,

$$\|X_{n+1}\|_{p\text{-var}, \tau_0} \leq \|H_{n+1}\|_{p\text{-var}, \tau_0} + \|Y_{n+1}\|_{p\text{-var}, \tau_0} \leq M.$$

By induction, (6) is satisfied for every  $n \in \mathbb{N}$ . Moreover,

$$\|Y_n\|_{1\text{-var}, \tau_0} \leq c_2 + c_1$$

and

$$\|H_n\|_{p\text{-var}, \tau_0} \leq 1$$

for every  $n \in \mathbb{N}^*$ .

For every  $t \in [0, T]$ , the map  $\|Z\|_{p\text{-var}, t, \cdot}$  is continuous from  $[t, T]$  into  $\mathbb{R}_+$  and  $\|Z\|_{p\text{-var}, t, t} = 0$ . Moreover, the constant  $m$  depends only on  $c_1$ ,  $p$ ,  $M$  and  $\|f\|_{\text{Lip}^\gamma}$ . So, since  $[0, T]$  is compact, there exists  $N \in \mathbb{N}^*$  and  $(\tau_k)_{k \in \llbracket 0, N \rrbracket} \in \mathfrak{D}_{[\tau_0, T]}$  such that

$$\|Z\|_{p\text{-var}, \tau_k, \tau_{k+1}} \leq m ; \forall k \in \llbracket 0, N-1 \rrbracket.$$

Since for every  $n \in \mathbb{N}^*$  the maps

$$\begin{aligned} (s, t) \in \Delta_T &\longmapsto \|X_n\|_{p\text{-var}, s, t}^p, \\ (s, t) \in \Delta_T &\longmapsto \|H_n\|_{p\text{-var}, s, t}^p \text{ and} \\ (s, t) \in \Delta_T &\longmapsto \|Y_n\|_{1\text{-var}, s, t} \end{aligned}$$

are control functions, recursively, the sequence  $(H_n, X_n, Y_n)_{n \in \mathbb{N}^*}$  is bounded in

$$\mathfrak{C}_T^{p,1} := C^{p\text{-var}}([0, T], \mathbb{R}^e) \times C^{p\text{-var}}([0, T], \mathbb{R}^e) \times C^{1\text{-var}}([0, T], \mathbb{R}^e).$$

By Proposition 2.6, for every  $n \in \mathbb{N}^*$  and  $(s, t) \in \Delta_T$ ,

$$\|H_n(t) - H_n(s)\| \leq c(p, p) \left( \|Df\|_\infty \sup_{n \in \mathbb{N}} \|X_n\|_{p\text{-var}, T} + \|f\|_\infty \right) \|Z\|_{p\text{-var}, s, t}.$$

Since  $(s, t) \in \Delta_T \mapsto \|Z\|_{p\text{-var}, s, t}$  is a continuous map such that  $\|Z\|_{p\text{-var}, t, t} = 0$  for every  $t \in [0, T]$ ,  $(H_n)_{n \in \mathbb{N}^*}$  is equicontinuous. Therefore, by Arzela-Ascoli's theorem together with Proposition 2.5, there exists an extraction  $\varphi : \mathbb{N}^* \rightarrow \mathbb{N}^*$  such that  $(H_{\varphi(n)})_{n \in \mathbb{N}^*}$  converges uniformly to an element  $H$  of  $C^{p\text{-var}}([0, T], \mathbb{R}^e)$ .

Since  $(H_{\varphi(n)})_{n \in \mathbb{N}^*}$  converges uniformly to  $H$ , by Theorem 2.2,  $(X_{\varphi(n)}, Y_{\varphi(n)})_{n \in \mathbb{N}^*}$  converges uniformly to  $(X, Y) := (v_H, w_H)$ . So, for every  $t \in [0, T]$ ,

$$\begin{cases} X(t) = H(t) + Y(t) \\ -\frac{dDY}{d|DY|}(t) \in N_{C_H(t)}(Y(t)) \text{ } |DY|\text{-a.e. with } Y(0) = a, \end{cases}$$

and by Proposition 2.5,

$$X \in C^{p\text{-var}}([0, T], \mathbb{R}^e) \text{ and } Y \in C^{1\text{-var}}([0, T], \mathbb{R}^e).$$

Moreover, since  $(X_{\varphi(n)})_{n \in \mathbb{N}^*}$  converges uniformly to  $X$ , by Proposition 2.7,

$$\lim_{n \rightarrow \infty} \left\| H_{\varphi(n)} - \int_0^\cdot f(X(s)) dZ(s) \right\|_{\infty, T} = 0.$$

Therefore, since  $(H_{\varphi(n)})_{n \in \mathbb{N}^*}$  converges also to  $H$  in  $C^0([0, T], \mathbb{R}^e)$ ,

$$H(t) = \int_0^t f(X(s)) dZ(s) ; \forall t \in [0, T].$$

□

**Proposition 3.2.** *Under Assumption 1.1, if  $p = 1$ , then Problem (2) has a unique solution which belongs to  $C^{1\text{-var}}([0, T], \mathbb{R}^e)$ .*

*Proof.* Consider two solutions  $(X, Y)$  and  $(X^*, Y^*)$  of Problem (2) on  $[0, T]$ . Since the map  $\|Z\|_{1\text{-var}, 0, \cdot}$  is continuous from  $[0, T]$  into  $\mathbb{R}_+$ , and since  $\|Z\|_{1\text{-var}, 0, 0} = 0$ , there exists  $\tau_0 \in [0, T]$  such that

$$\|Z\|_{1\text{-var}, \tau_0} \leq M := \frac{1}{4\|f\|_{\text{Lip}^\gamma}}.$$

For every  $t \in [0, \tau_0]$ ,

$$\begin{aligned} \|X(t) - X^*(t)\|^2 &= \|H(t) - H^*(t)\|^2 + 2 \int_0^t \langle Y(s) - Y^*(s), d(Y - Y^*)(s) \rangle + \\ &\quad 2 \int_0^t \langle H(t) - H^*(t), d(Y - Y^*)(s) \rangle \\ &\leq m_1(\tau_0)^2 + 2m_2(t) + 2m_3(t) \end{aligned}$$

with  $m_1(\tau_0) := \|H - H^*\|_{\infty, \tau_0}$ ,

$$m_2(t) := \int_0^t \langle X(s) - X^*(s), d(Y - Y^*)(s) \rangle$$

and

$$m_3(t) := \int_0^t \langle H(t) - H^*(t) - (H(s) - H^*(s)), d(Y - Y^*)(s) \rangle.$$

Consider  $t \in [0, \tau_0]$ . By Cauchy-Schwarz's inequality:

$$\begin{aligned} \|H(t) - H^*(t)\| &\leq \int_0^{\tau_0} \|f(X(s)) - f(X^*(s))\| \cdot \|\dot{Z}(s)\| ds \\ &\leq \|f\|_{\text{Lip}^\gamma} \|X - X^*\|_{\infty, \tau_0} \|Z\|_{1\text{-var}, \tau_0}. \end{aligned}$$

So,

$$(7) \quad m_1(\tau_0) \leq \frac{1}{4} \|X - X^*\|_{\infty, \tau_0}.$$

Since the map  $x \in C(t) \mapsto N_{C(t)}(x)$  is monotone,  $m_2(t) \leq 0$ . By the integration by parts formula,

$$\begin{aligned} m_3(t) &= \int_0^t \langle Y(s) - Y^*(s), d(H - H^*)(s) \rangle \\ &= \int_0^t \langle X(s) - X^*(s) - H(s) - H^*(s), (f(X(s)) - f(X^*(s))) dZ(s) \rangle. \end{aligned}$$

So, by Cauchy-Schwarz's inequality and Inequality (7),

$$\begin{aligned} m_3(t) &\leq \|Df\|_{\infty} \|X - X^*\|_{\infty, \tau_0} (\|X - X^*\|_{\infty, \tau_0} + \|H - H^*\|_{\infty, \tau_0}) \|Z\|_{1\text{-var}, \tau_0} \\ &\leq \|f\|_{\text{Lip}^\gamma} \|Z\|_{1\text{-var}, \tau_0} (1 + \|f\|_{\text{Lip}^\gamma} \|Z\|_{1\text{-var}, \tau_0}) \|X - X^*\|_{\infty, \tau_0}^2 \\ &\leq 5/16 \|X - X^*\|_{\infty, \tau_0}^2. \end{aligned}$$

Therefore,

$$\|X - X^*\|_{\infty, \tau_0}^2 \leq \frac{11}{16} \|X - X^*\|_{\infty, \tau_0}^2.$$

Necessarily,  $(X, Y) = (X^*, Y^*)$  on  $[0, \tau_0]$ . □

## 4. A SWEEPING PROCESS PERTURBED BY A ROUGH SIGNAL

Throughout this section,  $p \in [2, 3[$  and there exists  $\mathbb{Z} : [0, T] \rightarrow \mathcal{M}_d(\mathbb{R})$  such that  $\mathbf{Z} := (Z, \mathbb{Z}) \in G\Omega_{p,T}(\mathbb{R}^d)$ .

The existence of solutions to Problem (2) is established in Theorem 4.1.

**Theorem 4.1.** *Under Assumption 1.1, Problem (2) has at least one solution which belongs to  $C^{p\text{-var}}([0, T], \mathbb{R}^e)$ .*

*Proof.* Consider  $c_1 := 1/r$ ,

$$c_2 := c_1 \sup_{s \in [0, T]} \|\gamma(s)\|^2$$

and the discrete scheme

$$(8) \quad \begin{cases} X_n(t) = H_n(t) + Y_n(t) \\ H_n(t) = \int_0^t f(X_{n-1}(s)) d\mathbf{Z}(s) \\ -\frac{dDY_n}{d|DY_n|}(t) \in N_{C_{H_n}(t)}(Y_n(t)) \text{ } |DY_n|\text{-a.e. with } Y_n(0) = a \end{cases}$$

for Problem (2), initialized by

$$(9) \quad \begin{cases} -\frac{dDX_0}{d|DX_0|}(t) \in N_{C(t)}(X_0(t)) \text{ } |DX_0|\text{-a.e.} \\ X_0(0) = a. \end{cases}$$

Since the map  $\omega_{\mathbf{Z}}(0, \cdot)$  is continuous from  $[0, T]$  into  $\mathbb{R}_+$ , and since  $\omega_{\mathbf{Z}}(0, 0) = 0$ , there exists  $\tau_0 \in [0, T]$  such that

$$\begin{aligned} \omega_{\mathbf{Z}}(0, \tau_0) &\leq \frac{1}{c(p, 1)^p \|f\|_{\text{Lip}^\gamma}^p (M_2 + 1)^p} \wedge \frac{1}{M_2^p} \wedge \\ &\quad \frac{1}{(1 + 2^{p/2-1}(M_2^p + M_3^{p/2}) + 2^{p/2}\|f\|_{\text{Lip}^\gamma}^{p/2} M_2^{p/2})^2} \wedge \\ &\quad \frac{1}{(c_4(p) \vee c_7(p))^p (1 + 2M_2 + M_2^2 + M_3)^p} \end{aligned}$$

with  $M_1 := c_2 + c_1$ ,  $M_2 := M_1 + 1$ ,  $M_3 := 2(c_3(p) \vee c_6(p))(M_1^p + 1)^{1/p}$ ,  $c_1 := 1/r$  and

$$c_2 := c_1 \sup_{s \in [0, T]} \|\gamma(s)\|^2.$$

The constants  $c_k(p)$  with  $k \in \{3, 4, 6, 7\}$  are defined in the sequel and depend only on  $p$  and  $\|f\|_{\text{Lip}^\gamma}$ .

First of all, let us control the solution of the discrete scheme for  $n \in \{0, 1\}$ :

- ( $n = 0$ ) By (9) together with Proposition 2.3:

$$\begin{aligned} \|X_0\|_{1\text{-var}, \tau_0} &\leq \sup_{s \in [0, \tau_0]} \mathfrak{l}(r, \|\gamma(s)\|) \\ &\leq M_2. \end{aligned}$$

- ( $n = 1$ ) Since  $X_0 \in C^{1\text{-var}}([0, T], \mathbb{R}^e)$ , by Proposition 2.6:

$$\begin{aligned} \|H_1\|_{p\text{-var}, \tau_0} &\leq c(p, 1) \omega_{\mathbf{Z}}(0, \tau_0)^{1/p} \|f\|_{\text{Lip}^\gamma} (M_2 + 1) \\ &\leq 1. \end{aligned}$$

Since  $Y_1 = w_{H_1}$ , by Proposition 2.3:

$$\begin{aligned} \|Y_1\|_{1\text{-var}, \tau_0} &\leq c_2 + c_1 \|H_1\|_{p\text{-var}, \tau_0}^2 \\ &\leq M_1. \end{aligned}$$

Therefore,

$$\|X_1\|_{p\text{-var}, \tau_0} \leq \|H_1\|_{p\text{-var}, \tau_0} + \|Y_1\|_{p\text{-var}, \tau_0} \leq M_2.$$

Let us show that for every  $n \in \mathbb{N} \setminus \{0, 1\}$ ,

$$(10) \quad (X_{n-1}, f(X_{n-2})) \in \mathfrak{D}_Z^{p/2}([0, \tau_0], \mathbb{R}^e)$$

and

$$(11) \quad \begin{cases} \|Y_n\|_{1\text{-var}, \tau_0} \leq M_1 \\ \|X_n\|_{p\text{-var}, \tau_0} \leq M_2 \\ \|R_{X_{n-1}}\|_{p/2\text{-var}, \tau_0} \leq M_3. \end{cases}$$

Put  $X'_1 := f(X_0)$ . For every  $(s, t) \in \Delta_{\tau_0}$ ,

$$\begin{aligned} R_{X_1}(s, t) &= X_1(s, t) - X'_1(s)Z(s, t) \\ &= Y_1(s, t) + \int_s^t f(X_0(u))dZ(u) - f(X_0(s))Z(s, t). \end{aligned}$$

By Young-Love's estimate (see Friz and Victoir [11, Theorem 6.8]), for every  $(s, t) \in \Delta_{\tau_0}$ ,

$$\|R_{X_1}(s, t)\| \leq \|Y_1\|_{p/2\text{-var}, s, t} + \frac{1}{1 - 2^{1-3/p}} \|f\|_{\text{Lip}^\gamma} \|Z\|_{p\text{-var}, \tau_0} \|X_0\|_{p/2\text{-var}, s, t}.$$

By super-additivity of the control functions  $\|Y_1\|_{p/2\text{-var}, \cdot}^{p/2}$  and  $\|X_0\|_{p/2\text{-var}, \cdot}^{p/2}$ , there exists a constant  $c_3(p) > 0$ , depending only on  $p$  and  $\|f\|_{\text{Lip}^\gamma}$ , such that

$$\begin{aligned} \|R_{X_1}\|_{p/2\text{-var}, \tau_0} &\leq c_3(p)(\|Y_1\|_{p/2\text{-var}, \tau_0}^{p/2} + \|Z\|_{p\text{-var}, \tau_0}^{p/2} \|X_0\|_{p/2\text{-var}, \tau_0}^{p/2})^{2/p} \\ &\leq 2c_3(p)(M_1^p + \omega_{\mathbf{Z}}(0, \tau_0)M_2^p)^{1/p}. \end{aligned}$$

Then,  $\|R_{X_1}\|_{p/2\text{-var}, \tau_0} \leq 2c_3(p)(M_1^p + 1)^{1/p} \leq M_3$  and

$$(X_1, f(X_0)) \in \mathfrak{D}_Z^{p/2}([0, \tau_0], \mathbb{R}^e).$$

So, the rough integral

$$H_2 := \int_0^\cdot f(X_1(s))d\mathbf{Z}(s)$$

is well defined. For every  $(s, t) \in \Delta_T$ ,

$$\begin{aligned} \|H_2(s, t)\| &\leq \|f\|_{\text{Lip}^\gamma} \|Z\|_{p\text{-var}, s, t} + \|f\|_{\text{Lip}^\gamma}^2 \|Z\|_{p/2\text{-var}, s, t} + \mathfrak{I}_{f, \mathbf{Z}, X_n}(s, t) \\ &\leq (\|f\|_{\text{Lip}^\gamma} \vee \|f\|_{\text{Lip}^\gamma}^2) \omega_{\mathbf{Z}}(s, t)^{1/p} + \\ &\quad c(p) \|f\|_{\text{Lip}^\gamma} (1 \vee \|f\|_{\text{Lip}^\gamma}) (\|X_0\|_{p\text{-var}, s, t} + \|X_1\|_{p\text{-var}, s, t} + \\ &\quad \|X_1\|_{p\text{-var}, s, t}^2 + \|R_{X_1}\|_{p/2\text{-var}, s, t}) \omega_{\mathbf{Z}}(s, t)^{1/p} \\ &\leq c_4(p)(1 + 2M_2 + M_2^2 + M_3) \omega_{\mathbf{Z}}(s, t)^{1/p} \end{aligned}$$

where,  $c_4(p) > 0$  is a constant depending only on  $p$  and  $\|f\|_{\text{Lip}^\gamma}$ . By super-additivity of the control function  $\omega_{\mathbf{Z}}$ :

$$\begin{aligned} \|H_2\|_{p\text{-var}, \tau_0} &\leq c_4(p)(1 + 2M_2 + M_2^2 + M_3) \omega_{\mathbf{Z}}(0, \tau_0)^{1/p} \\ &\leq 1. \end{aligned}$$

So, as in the proof of Theorem 3.1,

$$\begin{aligned} \|Y_2\|_{p\text{-var}, \tau_0} &\leq c_2 + c_1 \|H_2\|_{p\text{-var}, \tau_0}^2 \\ &\leq c_2 + c_1 = M_1 \end{aligned}$$

and

$$\begin{aligned} \|X_2\|_{p\text{-var}, \tau_0} &\leq \|H_2\|_{p\text{-var}, \tau_0} + \|Y_2\|_{p\text{-var}, \tau_0} \\ &\leq 1 + c_2 + c_1 = M_2. \end{aligned}$$

Therefore, Conditions (10)-(11) hold true for  $n = 2$ .

Assume that Conditions (10)-(11) hold true until  $n \in \mathbb{N} \setminus \{0, 1\}$  arbitrarily chosen. Put  $X'_n := f(X_{n-1})$ . For every  $(s, t) \in \Delta_{\tau_0}$ ,

$$\begin{aligned} R_{X_n}(s, t) &= X_n(s, t) - X'_n(s)Z(s, t) \\ &= Y_n(s, t) + \int_s^t f(X_{n-1}(u))d\mathbf{Z}(u) - f(X_{n-1}(s))Z(s, t). \end{aligned}$$

So, for every  $(s, t) \in \Delta_{\tau_0}$ ,

$$\begin{aligned} \|R_{X_n}(s, t)\| &\leq \|Y_n(s, t)\| + \|Df(X_{n-1}(s))f(X_{n-2}(s))\mathbb{Z}(s, t)\| + \mathfrak{I}_{f, \mathbf{Z}, X_{n-1}}(s, t) \\ &\leq \|Y_n\|_{p/2\text{-var}, s, t} + \|f\|_{\text{Lip}^\gamma}^2 \|\mathbb{Z}\|_{p/2\text{-var}, s, t} + \\ &\quad c(p)(\|Z\|_{p\text{-var}, s, t} \|R_{f(X_{n-1})}\|_{p/2\text{-var}, s, t} + \\ &\quad \|\mathbb{Z}\|_{p/2\text{-var}, s, t} \|Df(X_{n-1}(\cdot))f(X_{n-2})\|_{p\text{-var}, s, t}) \\ &\leq \|Y_n\|_{p/2\text{-var}, s, t} + \|f\|_{\text{Lip}^\gamma}^2 \|\mathbb{Z}\|_{p/2\text{-var}, s, t} + \\ &\quad c(p)\|f\|_{\text{Lip}^\gamma} (\|Z\|_{p\text{-var}, \tau_0} \omega_n(s, t)^{2/p} + c_5(n, \tau_0) \|\mathbb{Z}\|_{p/2\text{-var}, s, t}) \end{aligned}$$

where,

$$\begin{aligned} c_5(n, \tau_0) &:= \|f(X_{n-2})\|_{p\text{-var}, \tau_0} + \|f(X_{n-2})\|_{\infty, \tau_0} \|X_{n-1}\|_{p\text{-var}, \tau_0} \\ &\leq \|f\|_{\text{Lip}^\gamma} (\|X_{n-2}\|_{p\text{-var}, \tau_0} + \|X_{n-1}\|_{p\text{-var}, \tau_0}) \end{aligned}$$

and  $\omega_n : \Delta_{\tau_0} \rightarrow \mathbb{R}_+$  is the control function defined by

$$\omega_n(u, v) := 2^{p/2-1}(\|X_{n-1}\|_{p\text{-var}, u, v}^p + \|R_{X_{n-1}}\|_{p/2\text{-var}, u, v}^{p/2})$$

for every  $(u, v) \in \Delta_{\tau_0}$ . By super-additivity of the control functions

$$\|Y_n\|_{p/2\text{-var}, \cdot}^{p/2}, \|\mathbb{Z}\|_{p/2\text{-var}, \cdot}^{p/2} \text{ and } \omega_n,$$

there exists a constant  $c_6(p) > 0$ , depending only on  $p$  and  $\|f\|_{\text{Lip}^\gamma}$ , such that:

$$\begin{aligned} \|R_{X_n}\|_{p/2\text{-var}, \tau_0} &\leq c_6(p)(\|Y_n\|_{p/2\text{-var}, \tau_0}^{p/2} + \|\mathbb{Z}\|_{p/2\text{-var}, \tau_0}^{p/2} + \\ &\quad \|Z\|_{p\text{-var}, \tau_0}^{p/2} \omega_n(0, \tau_0) + c_5(n, \tau_0)^{p/2} \|\mathbb{Z}\|_{p/2\text{-var}, \tau_0}^{p/2})^{2/p} \\ &\leq 2c_6(p)(\|Y_n\|_{p/2\text{-var}, \tau_0}^p + \\ &\quad (1 + \omega_n(0, \tau_0) + c_5(n, \tau_0)^{p/2})^2 \omega_{\mathbf{Z}}(0, \tau_0))^{1/p} \\ &\leq 2c_6(p)(M_1^p + \\ &\quad (1 + 2^{p/2-1}(M_2^p + M_3^{p/2}) + 2^{p/2}\|f\|_{\text{Lip}^\gamma}^{p/2} M_2^{p/2})^2 \omega_{\mathbf{Z}}(0, \tau_0))^{1/p}. \end{aligned}$$

Then,  $\|R_{X_n}\|_{p/2\text{-var}, \tau_0} \leq 2c_6(p)(M_1^p + 1)^{1/p} \leq M_3$  and

$$(X_n, f(X_{n-1})) \in \mathfrak{D}_{\mathbf{Z}}^{p/2}([0, \tau_0], \mathbb{R}^e).$$

So, the rough integral

$$H_{n+1} := \int_0^\cdot f(X_n(s))d\mathbf{Z}(s)$$

is well defined. For every  $(s, t) \in \Delta_T$ ,

$$\begin{aligned} \|H_{n+1}(s, t)\| &\leq \|f\|_{\text{Lip}^\gamma} \|Z\|_{p\text{-var}, s, t} + \|f\|_{\text{Lip}^\gamma}^2 \|\mathbb{Z}\|_{p/2\text{-var}, s, t} + \mathfrak{I}_{f, \mathbf{Z}, X_n}(s, t) \\ &\leq (\|f\|_{\text{Lip}^\gamma} \vee \|f\|_{\text{Lip}^\gamma}^2) \omega_{\mathbf{Z}}(s, t)^{1/p} + \\ &\quad c(p)\|f\|_{\text{Lip}^\gamma} (1 \vee \|f\|_{\text{Lip}^\gamma}) (\|X_{n-1}\|_{p\text{-var}, s, t} + \|X_n\|_{p\text{-var}, s, t} + \\ &\quad \|X_n\|_{p\text{-var}, s, t}^2 + \|R_{X_n}\|_{p/2\text{-var}, s, t}) \omega_{\mathbf{Z}}(s, t)^{1/p} \\ &\leq c_7(p)(1 + 2M_2 + M_2^2 + M_3) \omega_{\mathbf{Z}}(s, t)^{1/p} \end{aligned}$$

where,  $c_7(p) > 0$  is a constant depending only on  $p$  and  $\|f\|_{\text{Lip}^\gamma}$ . By super-additivity of the control function  $\omega_{\mathbf{Z}}$ :

$$\begin{aligned} \|H_{n+1}\|_{p\text{-var}, \tau_0} &\leq c_7(p)(1 + 2M_2 + M_2^2 + M_3)\omega_{\mathbf{Z}}(0, \tau_0)^{1/p} \\ &\leq 1. \end{aligned}$$

So, as in the proof of Theorem 3.1,

$$\begin{aligned} \|Y_{n+1}\|_{p\text{-var}, \tau_0} &\leq c_2 + c_1\|H_{n+1}\|_{p\text{-var}, \tau_0}^2 \\ &\leq c_2 + c_1 = M_1 \end{aligned}$$

and

$$\begin{aligned} \|X_{n+1}\|_{p\text{-var}, \tau_0} &\leq \|H_{n+1}\|_{p\text{-var}, \tau_0} + \|Y_{n+1}\|_{p\text{-var}, \tau_0} \\ &\leq 1 + c_2 + c_1 = M_2. \end{aligned}$$

By induction, Conditions (10)-(11) are satisfied for every  $n \in \mathbb{N} \setminus \{0, 1\}$ . As in the proof of Theorem 3.1, the sequence  $(H_n, X_n, Y_n)_{n \in \mathbb{N} \setminus \{0, 1\}}$  is bounded in  $\mathfrak{C}_T^{p,1}$ . In addition, the sequence  $(R_{X_n})_{n \in \mathbb{N} \setminus \{0, 1\}}$  is bounded in  $C^{p/2\text{-var}}([0, T], \mathbb{R}^e)$ .

For every  $n \in \mathbb{N} \setminus \{0, 1\}$  and  $(s, t) \in \Delta_T$ ,

$$\begin{aligned} \|H_n(s, t)\| &\leq (\|f\|_{\text{Lip}^\gamma} \vee \|f\|_{\text{Lip}^\gamma}^2 + \\ &\quad c(p)\|f\|_{\text{Lip}^\gamma}(1 \vee \|f\|_{\text{Lip}^\gamma})(\sup_{n \in \mathbb{N}} \|X_{n-2}\|_{p\text{-var}, T} + \sup_{n \in \mathbb{N}} \|X_{n-1}\|_{p\text{-var}, T} + \\ &\quad \sup_{n \in \mathbb{N}} \|X_{n-1}\|_{p\text{-var}, T}^2 + \sup_{n \in \mathbb{N}} \|R_{X_{n-1}}\|_{p/2\text{-var}, T})\omega_{\mathbf{Z}}(s, t)^{1/p}. \end{aligned}$$

Since  $\omega_{\mathbf{Z}}$  is a control function,  $(H_n)_{n \in \mathbb{N} \setminus \{0, 1\}}$  is equicontinuous. Therefore, by Arzela-Ascoli's theorem together with Proposition 2.5, there exists an extraction  $\varphi : \mathbb{N} \setminus \{0, 1\} \rightarrow \mathbb{N} \setminus \{0, 1\}$  such that  $(H_{\varphi(n)})_{n \in \mathbb{N} \setminus \{0, 1\}}$  converges uniformly to an element  $H$  of  $C^{p\text{-var}}([0, T], \mathbb{R}^e)$ .

Since  $(H_{\varphi(n)})_{n \in \mathbb{N} \setminus \{0, 1\}}$  converges uniformly to  $H$ , by Theorem 2.2, the sequence  $(X_{\varphi(n)}, Y_{\varphi(n)})_{n \in \mathbb{N} \setminus \{0, 1\}}$  converges uniformly to  $(X, Y) := (v_H, w_H)$ . So, for every  $t \in [0, T]$ ,

$$\begin{cases} X(t) = H(t) + Y(t) \\ -\frac{dDY}{d|DY|}(t) \in N_{C_H(t)}(Y(t)) \text{ } |DY|\text{-a.e. with } Y(0) = a, \end{cases}$$

and by Proposition 2.5,

$$X \in C^{p\text{-var}}([0, T], \mathbb{R}^e) \text{ and } Y \in C^{1\text{-var}}([0, T], \mathbb{R}^e).$$

By putting  $X' := f(X)$ ,  $X'$  (resp.  $R_X$ ) is the uniform limit of  $(X'_{\varphi(n)})_{n \in \mathbb{N} \setminus \{0, 1\}}$  (resp.  $(R_{X_{\varphi(n)}})_{n \in \mathbb{N} \setminus \{0, 1\}}$ ). So, by Proposition 2.14:

$$\lim_{n \rightarrow \infty} \left\| H_{\varphi(n)} - \int_0^\cdot f(X(s))d\mathbf{Z}(s) \right\|_{\infty, T} = 0.$$

Therefore, since  $(H_{\varphi(n)})_{n \in \mathbb{N}^*}$  converges also to  $H$  in  $C^0([0, T], \mathbb{R}^e)$ ,

$$H(t) = \int_0^t f(X(s))d\mathbf{Z}(s) ; \forall t \in [0, T].$$

□

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